Entanglement quantification with Fisher information

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We show that the Fisher information associated with entanglement-assisted coding has a monotonic relationship with the logarithmic negativity, an important entanglement measure, for certain classes of continuous variable (CV) quantum states of practical significance. These are the two-mode squeezed states and the non-Gaussian states obtained from them by photon subtraction. This monotonic correspondence can be expressed analytically in the case of pure states. Numerical analysis shows that this relationship holds to a very good approximation even in the mixed state case of the photon-subtracted squeezed states. The Fisher information is evaluated by the CV Bell measurement in the limit of weak signal modulation. Our results suggest that the logarithmic negativity of certain sets of non-Gaussian mixed states can be experimentally accessed without homodyne tomography, leading to significant simplification of the experimental procedure.

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Some important quantum information processing protocols with continuous variable (CV) systems essentially require highly nonlinear processes beyond Gaussian operations. An important example is entanglement distillation for a two-mode squeezed state, which is impossible to perform with Gaussian local operations and classical communication only [1]. One promising way of realizing such non-Gaussian operations is to use measurement-induced nonlinearity, such as photon subtraction from squeezed states with linear optics and photon counters [2]. The simplest such operation, which de-Gaussifies a single-mode squeezed state via a single photon subtraction, has recently been demonstrated [4, 5, 6], where the photon subtraction was done using an on/off type photon detector, i.e. an avalanche photodiode.

Extensions of this scheme to bipartite states enable one to perform entanglement distillation [3]. The primary concern is then to quantify the degree of entanglement enhancement. Methods for quantifying the entanglement of bipartite pure states have been established. In practice, however, imperfections are inevitable, making the output states mixed. For mixed states, especially non-Gaussian states, the evaluation of the entanglement is highly non-trivial. Two of the most frequently used measures of entanglement for mixed states are the negativity and logarithmic negativity (LN) [7]. These are straightforwardly calculable even for the mixed states. The LN was used to predict the gain of the entanglement distillation with photon subtraction in practical settings with on/off detectors [8]. Also, in the laboratory, the negativity of non-Gaussian entangled states has been evaluated by homodyne tomography for a particular type of photon-subtracted squeezed state where the entangled state is decomposable into a product of single-mode states, which greatly simplifies the tomographic process [9]. In general, however, one needs a complete description of the state to compute the LN. It is a highly non-trivial task to reconstruct a multi-mode CV state via tomographic measurement without any a priori knowledge about the state (e.g. symmetry or decomposability of the state).

An indirect but practical approach is to examine the increase of the performance of concrete protocols in terms of their associated figures of merit such as the fidelity of teleportation [10], degree of Bell inequality violation [11] and the mutual information of entanglement-assisted coding [12]. These quantities can be evaluated using a homodyne-based measurement, such as the Bell measurement of the conjugate pair of quadratures $\hat{x}$ and $\hat{p}$ satisfying $[\hat{x}, \hat{p}] = i/2$. This measurement is easily performed and does not require full tomographic reconstruction of the entangled state.

On the other hand, these quantities are not always true entanglement measures since they depend not only on the amount of entanglement but also on external factors, such as the input state for teleportation, the signal power and measurement strategy for entanglement-assisted coding. It might then be sensible to consider the limit of making these external factors as small as possible for assessing the intrinsic effect of entanglement. In particular, the close relationship between the LN and the mutual information of entanglement-assisted coding with weak modulation has been pointed out in [8]. The mutual information in the limit of weak signal modulation is reduced to the Fisher information [14], which is a fundamental quantity in statistics [15].

This implies another operational meaning of the LN, in addition to the one shown in [13], i.e., the direct rela-

tionship between the LN and the entanglement cost under the positive-partial-transpose-preserving operations for certain classes of quantum states. The latter relies on the asymptotic manipulation scenarios, while the Fisher information can be directly accessed by a simple coding scheme.

In this paper, we discuss the relationship between the LN and the Fisher information of entanglement-assisted coding for important classes of CV states such as the two-mode squeezed states and the photon-subtracted squeezed states. We show that there is a direct connection between the LN and the Fisher information for these states. Moreover, a similar relationship is also found for some entangled qubit states.

Let us denote the two-mode squeezed state as

$$|r^{(2)}\rangle_{AB} = \sum_{n=0}^{\infty} \alpha_n |n\rangle_A |n\rangle_B, \qquad (1)$$

where $\alpha_n \equiv \sqrt{1-\lambda^2}\lambda^n$ with $0 \leq \lambda = \tanh r < 1$. The beam C (D) is tapped off from the beam A (B) with a beam splitter of transmittance $T \equiv 1 - R$,

$$|\psi\rangle_{ABCD} = \hat{V}_{AC}(\theta)\hat{V}_{BD}(\theta)|r^{(2)}\rangle_{AB}|0\rangle_{CD}, \qquad (2)$$

where $\hat{V}_{ij}(\theta) = \exp[\theta(\hat{a}_i^\dagger \hat{a}_j - \hat{a}_i \hat{a}_j^\dagger)]$ is the beam splitter operator, and $\tan\theta = \sqrt{(1-T)/T}$. Then the tapped beam C (D) is measured by a photon detector (Fig. 1).

$$\hat{\rho}^{(AB)} = \frac{\text{Tr}_{CD}[|\psi\rangle_{(ABCD)}\langle\psi|(\hat{\Pi}_C^S \otimes \hat{\Pi}_D^S)]}{P_{\text{det}}}, \qquad (3)$$

where $\hat{\Pi}_k^S$ is the POVM element corresponding to the event selection at the beam path k, and $P_{\text{det}} = \text{Tr}_{ABCD}[|\psi\rangle_{(ABCD)}\langle\psi|(\hat{\Pi}_C^S \otimes \hat{\Pi}_D^S)]$ is the success probability of the event selection. Assuming photon number resolving detectors, and when single photons are simultaneously detected at the beams C and D ($\hat{\Pi}^S = |1\rangle\langle1|$), the output state is projected onto the non-Gaussian pure state

$$|\psi_{NG}^{(1)}\rangle_{AB} = \frac{1}{\sqrt{P_{\text{det}}^{(1)}}} \sum_{n=0}^{\infty} \alpha_{n+1}\xi_{(n+1),1}^2 |n\rangle_A |n\rangle_B, \qquad (4)$$
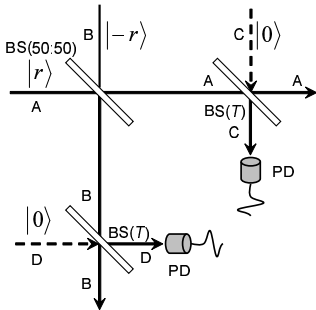
where $\xi_{nk} \equiv (-1)^k \binom{n}{k}^{1/2}(\sqrt{T})^{n-k}(\sqrt{R})^k$, with the binomial coefficient $\binom{n}{k}$, and

$$P_{\text{det}}^{(1)} = \frac{(1-\lambda^2)\lambda^2 T^2(1+\lambda^2 T^2)}{(1-\lambda^2 T^2)^3}\left(\frac{R}{T}\right)^2 \qquad (5)$$

The LN [7] of a quantum state $\hat{\rho}$ is defined by $E_\mathcal{N} = \log_2||\hat{\rho}^{PT}||$, where $||\cdot||$ denotes the trace norm, and PT means the partially transposition operation. For the pure state $|\chi\rangle_{AB} = \sum_n c_n|n\rangle_A|n\rangle_B$, in particular, the LN is expressed as $E_\mathcal{N}(|\chi\rangle) = 2\log_2(\sum_n c_n)$. For the two-mode squeezed state (1) and non-Gaussian pure state (4),

$$E_\mathcal{N}(|r^{(2)}\rangle) = \log_2\frac{1+\lambda}{1-\lambda} := E_\mathcal{N}^{SQ}, \qquad (6)$$

$$E_\mathcal{N}(|\psi_{NG}^{(1)}\rangle) = \log_2\frac{(1+\lambda T)^3}{(1+\lambda^2 T^2)(1-\lambda T)} := E_\mathcal{N}^{NG}, \quad (7)$$

respectively. In the range of

$$0 \leq \lambda \leq \lambda_{LN}^P = \frac{-1+T+\sqrt{-7T^2+18T-7}}{2T(2-T)}, \qquad (8)$$

$E_\mathcal{N}^{SQ} \leq E_\mathcal{N}^{NG}$ holds. In the limit as $T \to 1$, the non-Gaussian state $|\psi_{NG}^{(1)}\rangle$ always has higher entanglement, as quantified by the LN, than that of the two-mode squeezed state because $\lambda_{LN}^P \to 1$.

Let us now introduce the scheme for entanglement-assisted coding shown in Fig. 2 [8, 12]. Initially, the sender Alice and the receiver Bob share an entangled state $\hat{\rho}_{AB}^{(E)}$. Alice then encodes the quaternary phase-shift keyed (QPSK) signals $\{a_k\}$ with equal likelihood $P(a_k) = 1/4$ by the displacement operation on the beam A, $\hat{D}_A[(x_s+ip_s)/\sqrt{2}]$, with $x_s = \pm\sqrt{2}\beta$ and $p_s = \pm\sqrt{2}\beta$, where β is the signal amplitude. Bob decodes the signals by the CV Bell measurement $|\Pi(x,p)\rangle_{AB}$ and the quaternary decision on the measurement result (x,p) into an appropriate quadrant Q_l corresponding to the output b_l. The channel matrix is given by $P_{\beta'=\beta}^{(ch)}(b_l|a_k) = \iint_{Q_l} dxdp P(x,p|x_s^k,p_s^k)$, where $P(x,p|x_s,p_s) = {}_{AB}\langle\Pi(x,p)|\hat{U}_A(x_s,p_s)\hat{\rho}_{AB}^{(E)}\hat{U}_A^\dagger(x_s,p_s)|\Pi(x,p)\rangle_{AB}$ is the conditional probability.
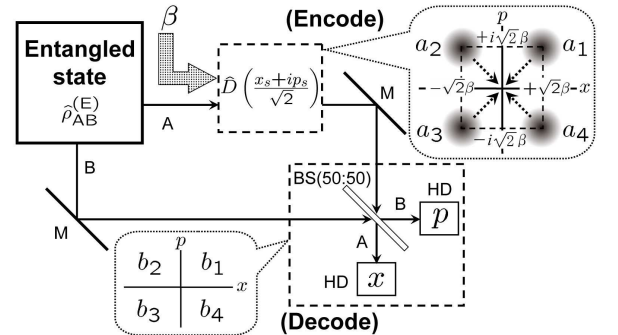

FIG. 1: Measurement-induced non-Gaussian operation on the two-mode squeezed state.

FIG. 2: Entanglement-assisted coding channel of QPSK.

The mutual information is defined by

$$I_\beta(\text{A};\text{B}) = \sum_a P(a) \sum_b P_\beta^{(\text{ch})}(b|a) \log_2 \frac{P_\beta^{(\text{ch})}(b|a)}{P(b)}, \quad (9)$$

where $P(b) = P_{\beta=0}^{(\text{ch})}(b|a)$. In the limit of small β $P_\beta^{(\text{ch})}(b|a)$ and $\log_2 P_\beta^{(\text{ch})}(b|a)$ can be expanded in a series around $\beta = 0$ as

$$I_\beta(\text{A};\text{B})$$
$$= \sum_a P(a) \sum_b \frac{\beta^2}{2\ln 2} \frac{1}{P_0^{(\text{ch})}(b|a)} \left[\frac{dP_{\beta'}^{(\text{ch})}(b|a)}{d\beta'} \bigg|_{\beta'=0} \right]^2$$
$$+ O(\beta^3). \quad (10)$$

Using the Fisher information for a parameter β' [15]

$$J_{\beta'} = \sum_b \frac{1}{P_{\beta'}^{(\text{ch})}(b|a)} \left\{ \frac{dP_{\beta'}^{(\text{ch})}(b|a)}{d\beta'} \right\}^2, \quad (11)$$

and taking the symmetry of the channel into account, we obtain

$$\lim_{\beta \to 0} \frac{I_\beta(\text{A};\text{B})}{\beta^2} = \frac{J_0}{2\ln 2}, \quad (12)$$

For the two-mode squeezed state (1) and the non-Gaussian pure state (4), the Fisher informations are

$$J_0^{\text{SQ}} = \frac{8}{\pi \ln 2} \frac{1+\lambda}{1-\lambda}. \quad (13)$$
$$J_0^{\text{NG}} = \frac{1}{2\pi \ln 2} \frac{(3\lambda^2 T^2 + 4\lambda T + 4)^2}{(1+\lambda^2 T^2)^2} \frac{1+\lambda T}{1-\lambda T}, \quad (14)$$

respectively. With Eqs. (6) and (7), we obtain

$$E_\mathcal{N}^{\text{SQ}} = \log_2 \frac{\pi \ln 2}{8} J_0^{\text{SQ}}, \quad (15)$$
$$E_\mathcal{N}^{\text{NG}} = \log_2 f(\lambda T) \frac{\pi \ln 2}{8} J_0^{\text{NG}}. \quad (16)$$

where

$$f(\lambda T) = \frac{16(1+\lambda T)^2(1+\lambda^2 T^2)}{(3\lambda^2 T^2 + 4\lambda T + 4)^2}, \quad (17)$$

which is very close to unity for $0 \le \lambda T < 1$. In Fig. 3, the correlation between J_0^{NG} and $E_\mathcal{N}^{\text{NG}}$ for various λT is indicated. In particular, for $0 \le \lambda T \lesssim 0.8$, the difference of $f(\lambda T)$ from unity is within $\pm 2\%$, showing that Eq. (15) and Eq. (16) are almost identical relations in a good approximation. Thus the Fisher information of the entanglement-assisted coding has monotonic correspondence with the LN for the two-mode squeezed states and the photon-subtracted squeezed states.

We then consider a more practical situation, where the photon subtraction is made by on/off type photon detectors, which is described by the set of POVM elements
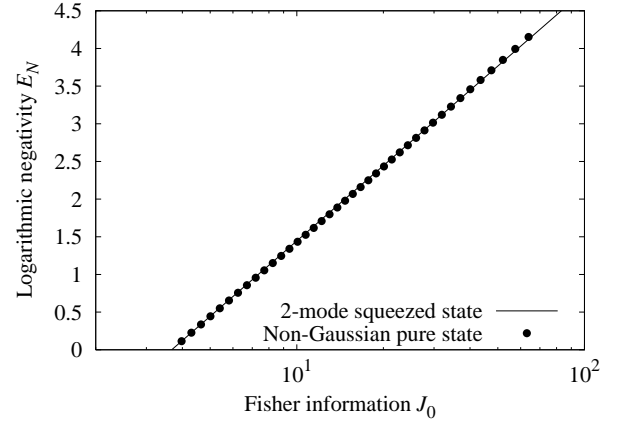

FIG. 3: Correlation between the Fisher information and logarithmic negativity for the two-mode squeezed states (solid line) and the photon-subtracted squeezed states (points) in various levels of a parameter λT (the combination of squeezing degree and beam splitter transmittance).

$\{\hat{\Pi}^{(\text{off})} = |0\rangle\langle 0|, \hat{\Pi}^{(\text{on})} = \hat{1} - |0\rangle\langle 0|\}$. In this case the generated state conditioned by the 'on' signal is generally a mixed state, as described by

$$\hat{\varrho}_{\text{NG}} = \frac{1}{\mathcal{P}_{\text{det}}} \sum_{i,j} |\Phi_{ij}\rangle_{(\text{AB})} \langle \Phi_{ij}|, \quad (18)$$

where $|\Phi_{ij}\rangle_{\text{AB}} = \sum_{n=\max\{i,j\}}^\infty \alpha_n \xi_{ni} \xi_{nj} |n-i\rangle_\text{A} |n-j\rangle_\text{B}$ and

$$\mathcal{P}_{\text{det}} = \frac{\lambda^2(1-T)^2(1+\lambda^2 T)}{(1-\lambda^2 T)(1-\lambda^2 T^2)}. \quad (19)$$

As the transmittance of tapping beam splitter get smaller, the mixedness of output state increases. The figure 4 illustrates the relationship between the Fisher information and LN for the photon-subtracted mixed states with various values of the transmittance. Here, the LN is calculated numerically using the methods described in [8]. We see here that, even for the mixed state cases, the same relation as Eq. (15) holds in a fairly good approximation. The accuracy of this approximation is within $\pm 2\%$ for $0 \le \lambda T \lesssim 0.8$, which is sufficient to verify the entanglement enhancement via the measurement-induced non-Gaussian operation. For $\lambda = 0.4$, for example, the Fisher information with the two-mode squeezed state is $J_0^{\text{SQ}} = 8.572$, and those of non-Gaussian pure and mixed states of photon-subtracted squeezed states with $T = 0.9$, are $J_0^{\text{NG(P)}} = 12.992$ and $J_0^{\text{NG(M)}} = 12.153$, respectively. The degrees of improvement are 51.6% and 41.8%, respectively, which are much greater than the accuracy of the approximation.

We finally show that the entanglement evaluation with the Fisher information is applicable to other entangled states. An interesting example is entangled photon-number-qubit state such as $\hat{\rho}_{\text{qubit}} = t|\xi\rangle\langle\xi| + (1-t)[|c_0|^2|0\rangle\langle 0| \otimes |1\rangle\langle 1| + |c_1|^2|1\rangle\langle 1| \otimes |0\rangle\langle 0|]$, where $|\xi\rangle =$
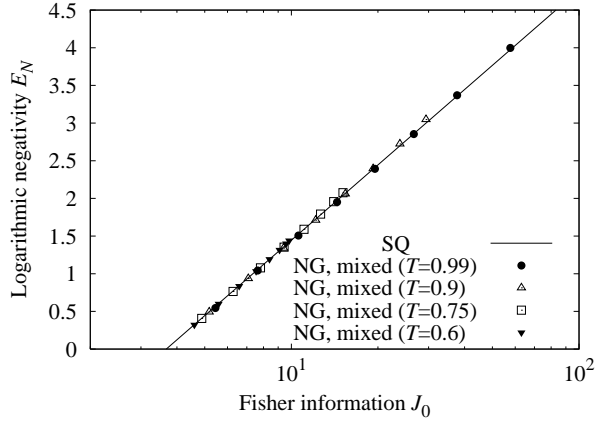
FIG. 4: Correlation between the Fisher information and logarithmic negativity for the two-mode squeezed state (solid line) and non-Gaussian mixed states of photon-subtracted squeezed state (variety of points).

$c_0|0\rangle|1\rangle + c_1|1\rangle|0\rangle$ and the parameter $0 \le t \le 1$ indicates its mixedness. The coefficients satisfy $|c_0|^2 + |c_1|^2 = 1$ and c_1 has the relative argument ϕ to c_0. Applying such a state to our entanglement-assisted coding scheme, the conditional probability $P_{\text{qubit}}(x, p|x_s, p_s)$ is obtained as a function of $t|c_0||c_1|$ and ϕ, and so is the Fisher information J_0^{qubit}. The dependency on ϕ is caused by our specific choice of QPSK encoding, and averaging J_0^{qubit} with respect to ϕ removes its arbitrariness. The average Fisher information $\bar{J}_0^{\text{qubit}} = \frac{1}{2\pi} \int_0^{2\pi} d\phi J_0^{\text{qubit}}$ is now a function of the unit $t|c_0||c_1|$ only. On the other hand, the LN for $\hat{\rho}_{\text{qubit}}$ is evaluated with $E_{\mathcal{N}}^{\text{qubit}} = \log_2(1 + 2t|c_0||c_1|)$. Therefore, it is also the case here that $\bar{J}_0^{\text{qubit}}$ and $E_{\mathcal{N}}^{\text{qubit}}$ are directly related, although this is a different relationship from that expressed by Eqs. (15) and (16).

For the other entangled photon-number-qubit states such as $|\xi'\rangle = c_0'|0\rangle|0\rangle + c_1'|1\rangle|1\rangle$ and their associated mixed states, one cannot find the one-to-one correspondence between the Fisher information and the LN any longer in our scheme. However, $|\xi'\rangle$ can be transformed into $|\xi\rangle$ with the local flipping of $0 \leftrightarrow 1$. This operation can, with current technology, be performed unitarily to arbitrarily high accuracy and thus in a way which preserves entanglement [16, 17]. This means that, by modifying the decoding strategy, inserting the flipping operation, the Fisher information can turn to work again. These two examples imply that, under a suitable encoding and decoding strategy, the correspondence between the Fisher information and the LN might always appear for arbitrary bipartite entangled state.

In this paper, we showed a direct relationship between the logarithmic negativity and the Fisher information of

the entanglement-assisted coding for important classes of CV states, i.e. the two-mode squeezed states and the photon-subtracted squeezed states, and also entangled qubit states. This is the first observation of direct connections between the logarithmic negativity and the Fisher information, and we have found that they hold for a wide class of bipartite entangled states. It is an important future goal to generalize this to a wider class of states by optimizing the encoding and decoding strategies.

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